LECTURE #15

CS 170

Spring 2021

Last time:

Maximizing flow in a network (graph with capacities)

Max Flow reduces to Linear Programming

Direct algorithm based on augmenting paths in residual network

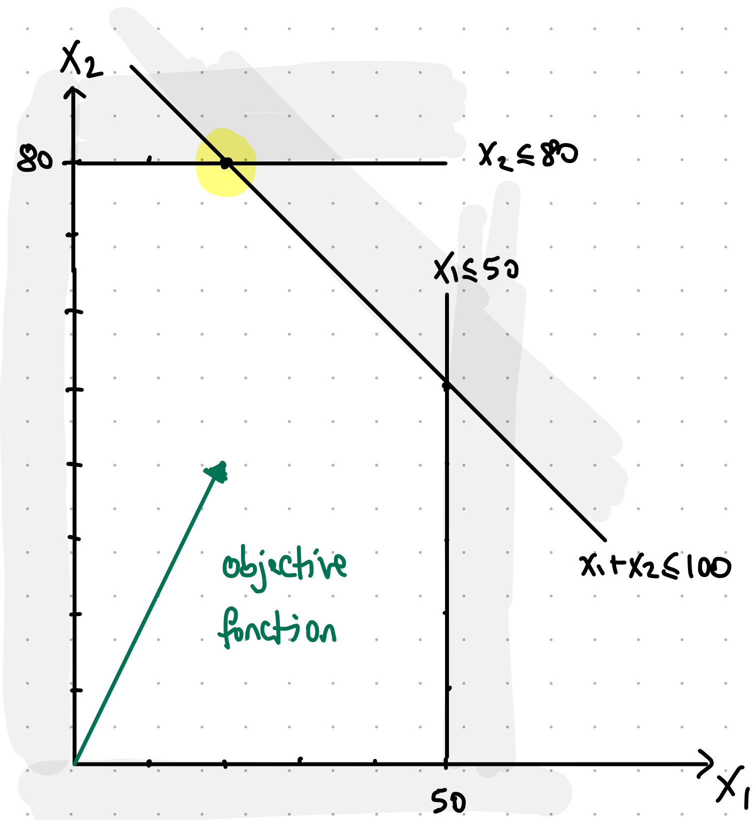
Max-Flow Min-Cut Theorem: $\boxed{\max_{\text{flow } f} \text{val}(f) = \min_{\text{cut } (L,R)} \text{capacity}(L,R)}$

this is an example of duality

Today:

Duality in Linear Programming

Recall example from LP lecture:



$$\max \quad 2x_1 + 4x_2 \quad \text{s.t.} \quad \begin{array}{l} x_1, x_2 \geq 0 \\ x_1 \leq 50,\ x_2 \leq 80 \\ x_1 + x_2 \leq 100 \end{array}$$

The solution was $(x_1, x_2) = (20, 80)$

with value $2x_1 + 4x_2 = 2 \cdot 20 + 4 \cdot 80 = 360$.

We can prove that the solution is optimal:

$$\begin{array}{rll} & 0 \cdot & (x_1 \leq 50) \\ + & 2 \cdot & (x_2 \leq 80) \\ + & 2 \cdot & (x_1 + x_2 \leq 100) \\ \hline & & 2x_1 + 4x_2 \leq 360 \end{array}$$

Where does this optimality certificate come from?

A more systematic approach: introduce variables y_1, y_2, y_3 and consider

$$\begin{array}{rl} & y_1 \cdot (x_1 \leq 50) \\ + & y_2 \cdot (x_2 \leq 80) \\ + & y_3 \cdot (x_1 + x_2 \leq 100) \\ \hline \end{array}$$

& $y_1, y_2, y_3 \geq 0$ (or else inequalities flip)

From prior slide:

max $2x_1 + 4x_2$ s.t. $x_1, x_2 \geq 0$

$x_1 \leq 50,\ x_2 \leq 80$

$x_1 + x_2 \leq 100$

$$\underbrace{(y_1 + y_3)}_{=2} x_1 + \underbrace{(y_2 + y_3)}_{=4} x_2 \leq 50 y_1 + 80 y_2 + 100 y_3$$

$=2$ $\quad = 4$ ← to match objective function

≥ 2 $\quad \geq 4$ ← also ok

Hence: $2x_1 + 4x_2 \leq \underbrace{50 y_1 + 80 y_2 + 100 y_3}_{\text{want to MINIMIZE for best upper bound}}$ if $\left\{ \begin{array}{l} y_1, y_2, y_3 \geq 0 \\ y_1 + y_3 \geq 2 \\ y_2 + y_3 \geq 4 \end{array} \right\}$.

In sum the dual LP to the original (primal) LP is:

$$\boxed{\begin{array}{ll} \text{min} & 50y_1 + 80y_2 + 100y_3 \\ \text{s.t.} & y_1, y_2, y_3 \geq 0 \\ & y_1 + y_3 \geq 2 \\ & y_2 + y_3 \geq 4 \end{array}}$$

If we have solutions to primal LP and to dual LP that have same value then they are both optimal.

Eg $(x_1, x_2) = (20, 80)$ & $(y_1, y_2, y_3) = (0, 2, 2)$ both have 360.

The general case

$k = \#$ variables $\quad m = \#$constraints $= |I| + |E|$ (N is separate)

- **Primal** $\quad \max \; c_1 x_1 + \cdots + c_k x_k$ s.t.

$$\begin{aligned} &\forall i \in I \quad a_{i1} x_1 + \cdots + a_{ik} x_k \leq b_i \\ &\forall i \in E \quad a_{i1} x_1 + \cdots + a_{ik} x_k = b_i \\ &\forall j \in N \quad x_j \geq 0 \end{aligned}$$

- **Transformation:**

$$\forall i \in I \quad y_i \cdot (a_{i1} x_1 + \cdots + a_{ik} x_k \leq b_i) \quad \text{with } y_i \geq 0$$

$$\forall i \in E \quad y_i \cdot (a_{i1} x_1 + \cdots + a_{ik} x_k = b_i) \quad \text{with } y_i \text{ anything}$$

$$\sum_{j \in N} \underbrace{\left(\sum_{i \in I \cup E} a_{ij} y_i \right)}_{\geq c_j} x_j + \sum_{j \notin N} \underbrace{\left(\sum_{i \in I \cup E} a_{ij} y_i \right)}_{= c_j} x_j \leq \underbrace{\sum_{i \in I} b_i y_i + \sum_{i \in E} b_i y_i}_{\text{MINIMIZE}}$$

WANT: $\geq c_j$ (because $x_j \geq 0$) $\quad = c_j$ (because x_j can be neg) $\quad$ MINIMIZE (best upper bound)

In sum:

- 1 multiplier variable per primal constraint
- 1 constraint per primal variable (coefficient $\geq$ or $=$)
- minimize the induced RHS of the sum

$m = \#\text{ variables} = |I| + |E|$ $k = \#\text{ constraints}$ (I is separate)

- DUAL: $\min\ b_1 y_1 + \cdots + b_m y_m$ s.t. $\forall j \in N \quad a_{1j} y_1 + \cdots + a_{mj} y_m \geq c_j$

  $\forall j \notin N \quad a_{1j} y_1 + \cdots + a_{mj} y_m = c_j$

  $\forall i \in I \quad y_i \geq 0$

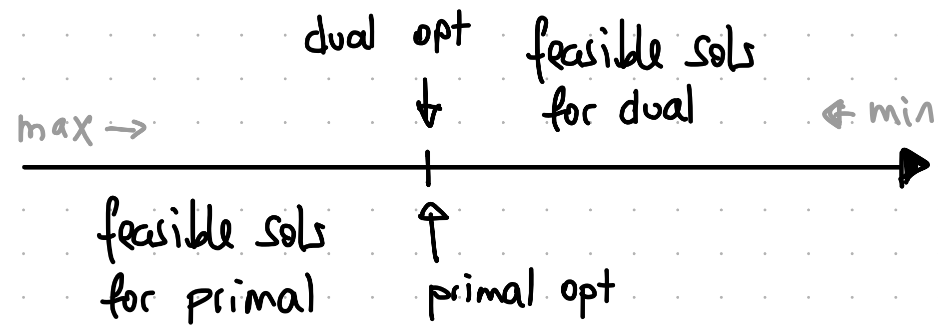
A notable special case:
- all constraints are inequalities
- all variables are non-negative

In this case the dual is easy to summarize:

Primal LP	Dual LP
$\max \langle c, x \rangle$	$\min \langle b, y \rangle$
s.t. $Ax \leq b$	s.t. $A^T y \geq c$
$x \geq 0$	$y \geq 0$

DUALITY THEOREM

If the primal LP has a bounded max, then the dual LP has a bounded min, and the two values are equal.



We will not prove the (strong) duality theorem.

Visualizing Duality

Compute the shortest path from s to t in an undirected graph G w/ positive lengths ℓ.

$\Rightarrow$ We can use Dijkstra's algorithm.

An alternative:

Physical model where V balls are connected according to E via strings, and each string from u to v has length $\ell(u,v)$.
The shortest path from s to t is found as follows:

pull s away from t until no longer possible

We are maximizing distance, while "shortest path" is about minimization. (!)

Why? We are solving the dual LP of shortest paths.

$$\boxed{\begin{array}{l} \max \quad x_t \quad \text{s.t.} \\ \bullet\ x_s = 0 \\ \bullet\ \forall (u,v) \in E:\ x_v \leq x_u + \ell(u,v) \end{array}}$$

$\leftarrow$ claim: optimum x_t^* is s.t. $x_t^* = \text{dist}(s,t)$

$x_t^* \geq \text{dist}(s,t)$: the constraints are satisfied by $\forall v \in V \ \ x_v := \text{dist}(s,v)$.

$x_t^* \leq \text{dist}(s,t)$: let $(s, v_1, \ldots, v_k, t)$ be a path with length $\text{dist}(s,t)$.

Then $x_s = 0 \wedge x_{v_1} \leq x_s + \ell(s,v_1) \wedge x_{v_2} \leq x_{v_1} + \ell(v_1,v_2) \wedge \ldots \Rightarrow x_t \leq \ell(s,v_1) + \ell(v_1,v_2) + \ldots = \text{dist}(s,t)$.